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Pseudospin Symmetry for a Ring-Shaped Non-spherical Harmonic Oscillator Potential

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Abstract The pseudospin symmetry for a ring-shaped non-spherical harmonic oscillator potential is investigated by solving the Dirac equation with equal mixture of scalar and vector potentials with opposite signs. The normalized spinor wave function and energy equation are obtained, the algebraic property of the energy equation and some particular cases are also discussed.

Keywords Ring-shaped non-spherical harmonic oscillator potential · Pseudospin symmetry · Dirac equation

1 Introduction

Nearly forty years ago, the pseudospin symmetry was introduced into nuclear physics to accommodate an observed near degeneracy of certain normal-parity shell-model orbitals with nonrelativistic quantum numbers $(n_r, \ell, j = \ell + 1/2)$ and $(n_r - 1, \ell + 2, j = \ell + 3/2)$, where n_r, ℓ and j are the single-nucleon radial, orbital, and total angular momentum quantum numbers, respectively [1, 2]. This doublet structure is expressed in terms of a "pseudo-orbital" angular momentum $\tilde{\ell} = \ell + 1$, coupled to a "pseudospin" $\tilde{s} = 1/2$ with $j = \tilde{\ell} \pm \tilde{s}$. For example, the shell model orbitals $[n_r s_{1/2}, (n_r - 1)d_{3/2}]$ will have $\tilde{\ell} = 1$, $[n_r p_{3/2}, (n_r - 1) f_{5/2}]$ will have $\tilde{\ell} = 2$, etc. These doublets are almost degenerate with respect to pseudospin, since $j = \tilde{\ell} \pm \tilde{s}$ for the two states in the doublet [3]. Then the single-particle energy is approximately independent of the orientation of the pseudospin leading to an approximate pseudospin symmetry. However, since the suggestion of pseudospin symmetry is made, the origin of this symmetry has eluded explanation until Ginocchio revealed that the pseudo-orbital angular momentum is nothing but the "orbital angular momentum" of the lower component of Dirac spinor and built the connection between pseudospin symmetry and the equality in magnitude but difference in sign of the scalar potential $V_s(\vec{r})$ and vector potential

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$V_V(\vec{r})$ [4, 5]. The pseudospin symmetry has been used to explain a number of phenomena in nuclear structure including superdeformation [6] and identical bands [7–9]. Recently, for any spin–orbit quantum number k, the quasi-analytical solutions of the Dirac equation with pseudospin symmetry have been presented for the Eckart potential by using the asymptotic iteration method [10], and for the Morse potential by using the Nikiforov–Uvarov method [11], the exact quantization rule method [12] and the asymptotic iteration method [13].

As we know, the harmonic oscillator potential is one of the exactly solvable potentials in quantum mechanics and the relativistic harmonic oscillator potential has received considerable attention in recent years. Chen et al. [14], using a Dirac Hamiltonian with scalar S and vector V potentials quadratic in space coordinates, found a harmonic-oscillator like second order equation which can be solved analytically for $\Delta = V - S = 0$, as considered before by Kukulin [15], and also for $\Sigma = V + S = 0$. Ginocchio has solved the triaxial, axial and spherical harmonic oscillators for the case $\Delta = 0$ and applied it to the study of antinucleons embedded in nuclei [16, 17]. The case $\Sigma = 0$ is particularly relevant in nuclear physics, since it is usually pointed out as a necessary condition for occurrence of pseudospin symmetry in nuclei [18]. However, considering the fact that the realistic nuclei often deviate from the harmonic oscillator models, several other oscillator models, such as the non-spherical harmonic oscillator (NHO) [19], the ring-shaped harmonic oscillator (RHO) [20, 21], the double ring-shaped harmonic oscillator (DRHO) [22, 23] and the Dirac oscillator [24, 25] have been introduced and studied in the framework of both nonrelativistic and relativistic. In this domain Dong et al. have presented a ring-shaped non-spherical harmonic oscillator (RNHO) potential, in spherical coordinates which is defined as [26]

$$V(r,\theta) = \frac{1}{2}M\omega^2 r^2 + \frac{\hbar^2\alpha}{2Mr^2} + \frac{\hbar^2\beta}{2Mr^2\sin^2\theta}, \tag{1}$$

and obtained the nonrelativistic energy spectra and wave functions by solving the corresponding Schrödinger equation, and established the creation and annihilation operators for the radial wave functions with the factorization method. It is shown that these operators satisfy a SU(1, 1) dynamic group. Following this work, Dong et al. have put forward a new anharmonic oscillator potential, or another RNHO potential, it has the form [27]

$$V(r,\theta) = \frac{1}{2}M\omega^2 r^2 + \frac{\hbar^2 A}{2Mr^2} + \frac{\hbar^2 B\cos^2\theta}{2Mr^2\sin^2\theta}, \tag{2}$$

and studied this potential in the framework above, and showed that this new anharmonic oscillator potential possesses a hidden symmetry between $E(r)$ and $E(ir)$ by substituting $r \to ir$. Guo et al. have solved the Dirac equation with the RNHO potential (see (1)) under the condition of pseudospin symmetry and several particular cases are also discussed [28]. Recently, we have proposed a new RNHO potential, which is replaced the term $(\sin^{-2}\theta\cos^2\theta)$ by $(\sin^{-2}\theta\cos\theta)$ in (2), and solved the Klein-Gordon equation and Dirac equation with equal scalar and vector potentials, and obtained energy spectrum of bound states and wave functions [29]. Considering that there has been a wide interest in relativistic potentials involving mixtures of vector and scalar potentials with opposite signs, the interest lies on attempts to explain the pseudospin symmetry. In this study, we will solve the Dirac equation with this new RNHO potential under the condition of pseudospin symmetry, i.e., $\Sigma = 0$ and give the spinor wave function and energy spectrum for bound states. Since the harmonic oscillator as an ideal model potential describing between the atoms in molecules is proved to be anharmonic in practice, this new RNHO model may be interpreted in such a way that the second term describes the dipole-like interaction and the third represents the angular-dependent interaction between the atoms apart from the principal harmonic oscillator interaction term.

The paper is organized as follows. In Sect. 2 the Dirac equation with this system is separated into the angular and radial components, and each of them is solved analytically. The exact normalized spinor wave functions and the energy spectra for bound states are obtained. Section 3 is devoted to discuss the pseudospin symmetry of the Dirac-RNHO problem and the algebraic property of the energy equation, showing that with the given quantum numbers n_r and L, only discrete, positive energies are allowed for this RNHO potential. Moreover, the studies of another kind of non-central potential, i.e. the Hartmann potential are reviewed. Some concluding remarks are given in Sect. 4. In spherical coordinates, the RNHO oscillator potential is defined as

$$V(r,\theta)=\frac{1}{2}M\omega^2r^2+\frac{\hbar^2\eta}{2Mr^2}+\frac{\hbar^2\gamma\cos\theta}{2Mr^2\sin^2\theta}, \tag{3}$$

where M and ω are the rest mass and frequency of the oscillator, η and γ are the dimensionless parameters, respectively. For simplicity, $\hbar=c=\omega=1$ is adopted in following calculation.

2 Solutions of Dirac Equation with the New RNHO Potential

The Dirac equation for a nucleon with rest mass M moving in an attractive scalar potential $S(\vec{r})$ and a repulsive vector potential $V(\vec{r})$ can be written as

$$[\vec{\alpha}\cdot\vec{\mathrm{p}}+\beta(M+S(\vec{r}))]\psi(\vec{r})=[\varepsilon-V(\vec{r})]\psi(\vec{r}), \tag{4}$$

where $\vec{\mathrm{p}}$ is the momentum operator and M is the fermion mass, and $\vec{\alpha}$ and β are 4×4 matrices which, in the usual representation, take the form

$$\vec{\alpha}=\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix},\qquad \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}. \tag{5}$$

Here $\vec{\sigma}$ is the vector Pauli spin matrix and in Pauli-Dirac representation. Let

$$\psi(\vec{r})=\begin{pmatrix}\varphi(\vec{r})\\ \chi(\vec{r})\end{pmatrix}, \tag{6}$$

and

$$\vec{\sigma}\cdot\vec{\mathrm{p}}\chi(\vec{r})=[\varepsilon-M-\Sigma]\varphi(\vec{r}), \tag{7a}$$

$$\vec{\sigma}\cdot\vec{\mathrm{p}}\varphi(\vec{r})=[\varepsilon+M-\Delta]\chi(\vec{r}), \tag{7b}$$

where $\Sigma=V(\vec{r})+S(\vec{r})$, $\Delta=V(\vec{r})-S(\vec{r})$. Under the condition of pseudospin symmetry $(\Sigma=0)$, (7) become

$$\varphi(\vec{r})=\frac{\vec{\sigma}\cdot\vec{\mathrm{p}}}{\varepsilon-M}\chi(\vec{r}), \tag{8a}$$

$$\vec{\sigma}\cdot\vec{\mathrm{p}}\varphi(\vec{r})=[\varepsilon+M-\Delta]\chi(\vec{r}). \tag{8b}$$

Substituting (8a) into (8b) and taking Δ to be the RNHO potential, a Schrödinger-like equation is obtained for the low component as

$$\left[\vec{\mathrm{p}}^2 - (\varepsilon^2 - M^2) + (\varepsilon - M)\left(\frac{1}{2}Mr^2 + \frac{\eta}{2Mr^2} + \frac{\gamma\cos\theta}{2Mr^2\sin^2\theta}\right)\right]\chi(\vec{r}) = 0. \tag{9}$$

From (9), it can be seen that, as the decoupling of pseudospin and pseudo-orbital momentum, the low component spinor can be just a spin up or spin down spinor, i.e., $\binom{1}{0}$ or $\binom{0}{1}$, multiplied by a function of the spherical coordinates, i.e., $\chi(\vec{r})$ can be written as [28]

$$\chi(r,\theta,\phi) = r^{-1}u(r)H(\theta)K(\phi)\tilde{\chi}_m, \tag{10}$$

in which m is $\pm 1/2$ and $\tilde{\chi}_m$ is a spin up or spin down two-component spinor. Substituting (10) into (9), the differential equations for the radial and angular parts are separated as

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}u(r) + \left[(\varepsilon^2 - M^2) - \frac{1}{2}(\varepsilon - M)Mr^2 - \frac{\lambda + (\frac{\varepsilon - M}{2M})\eta}{r^2}\right]u(r) = 0, \tag{11}$$

$$\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right)H(\theta) + \left[\lambda\sin^2\theta - \frac{1}{2}\left(\frac{\varepsilon - M}{M}\right)\gamma\cos\theta - \Lambda^2\right]H(\theta) = 0, \tag{12}$$

$$\frac{\mathrm{d}^2K(\phi)}{\mathrm{d}\phi^2} + \Lambda^2K(\phi) = 0. \tag{13}$$

Where Λ and λ are separation constants. The periodic boundary conditions of (13) is given by $K(\phi + 2\pi) = K(\phi)$. Then the solution of (13) can be obtained immediately as

$$K_\Lambda(\phi) = \frac{1}{\sqrt{2\pi}}\exp(i\Lambda\phi), \quad \Lambda = 0, \pm 1, \pm 2\ldots. \tag{14}$$

In order to get the solution of (12), a new variable is introduced

$$x = \frac{1}{2}(1 + \cos\theta) = \cos^2\frac{\theta}{2}. \tag{15}$$

Substituting (15) into (12) yields

$$x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}H(x) + [\lambda x(1-x) - p^2 - (q^2 - p^2)x]H(x) = 0. \tag{16}$$

Where

$$p = \frac{1}{2}\left|\Lambda^2 - \frac{(\varepsilon - M)\gamma}{2M}\right|^{\frac{1}{2}}, \qquad q = \frac{1}{2}\left|\Lambda^2 + \frac{(\varepsilon - M)\gamma}{2M}\right|^{\frac{1}{2}}. \tag{17}$$

The physically acceptable solution of (16) could be expressed as

$$H(x) = x^p(1-x)^q g(x). \tag{18}$$

Substituting (18) into (16), we get

$$x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}g(x) + [k - (i + j + 1)x]\frac{\mathrm{d}}{\mathrm{d}x}g(x) - (i \times j)g(x) = 0. \tag{19}$$

Equation (19) is a hypergeometric equation, and its solution is a hypergeometric function

$$g(x) = \mathrm{F}(i, j, k, x), \tag{20}$$

in which

$$\begin{cases} i = p + q - \ell, \\ j = p + q + 1 + \ell, \\ k = 2p + 1, \end{cases} \tag{21}$$

$$\lambda = \ell(\ell + 1). \tag{22}$$

However, the boundary conditions satisfied by the θ angular wave function demands that the hypergeometric function must be terminated as a polynomial, i.e.

$$i = p + q - \ell = -n_3 \quad (n_3 = 0, 1, 2, \ldots), \tag{23}$$

where $p + q$ and ℓ do not need to be integers and that only their difference must be integers. According to the Ref. [30], n_3 corresponds to the number of quanta for oscillations along the symmetry axis. Finally, we get the normalized solutions of θ angular equation (12) as

$$H_{n_3}(\theta) = N_{n_3} \cos^{2p}\frac{\theta}{2} \sin^{2q}\frac{\theta}{2} F\left(-n_3, p + q + 1 + \ell, 2p + 1, \cos^2\frac{\theta}{2}\right), \tag{24}$$

where N_{n_3} is the normalization constant. By using the following relations [31]

$$\begin{gathered} \mathrm{j}_n(\alpha, \gamma, x) = \mathrm{F}(-n, \alpha + n, \gamma, x), \\ \int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma} \mathrm{j}_n(\alpha, \gamma, x)\mathrm{j}_k(\alpha, \gamma, x)\mathrm{d}x = \frac{\Gamma^2(\gamma)\Gamma(\alpha + n - \gamma + 1)}{\Gamma(\alpha + n)\Gamma(\gamma + n)} \frac{n!}{(\alpha + 2n)} \delta_{nk}, \end{gathered} \tag{25}$$

and the orthogonality of the angular wave functions

$$\int_{-1}^{1} [H_n(x)]^2 \mathrm{d}x = 1, \tag{26}$$

the normalization constant is determinated as

$$N_{n_3} = \frac{1}{(2p)!}\sqrt{\frac{(2p + 2q + n_3)!(2p + n_3)!(2p + 2q + 2n_3 + 1)}{2n_3!(2q + n_3)!}}. \tag{27}$$

For the radial equation (11), a new variable $\rho = Br$ is introduced. By setting

$$\begin{gathered} B = \left[\left(\frac{\varepsilon - M}{2}\right)M\right]^{\frac{1}{4}}, \qquad A = (\varepsilon + M)\sqrt{\frac{2(\varepsilon - M)}{M}}, \\ L(L + 1) = \ell(\ell + 1) + \frac{(\varepsilon - M)\eta}{2M}. \end{gathered} \tag{28}$$

Equation (11) can be rearranged as

$$\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}u(\rho) + \left[A - \rho^2 - \frac{L(L + 1)}{\rho^2}\right]u(\rho) = 0. \tag{29}$$

The radial equation (29) has an irregular singularity at $\rho = \infty$. Furthermore, it has a singularity at $\rho = 0$. Therefore, it is reasonable to set

$$u(\rho) = \rho^{L+1} e^{-\frac{\rho^2}{2}} f(\rho). \tag{30}$$

Substituting (30) into (29), we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} f(\rho) + \left[\frac{2(L+1)}{\rho} - 2\rho\right] \frac{\mathrm{d}}{\mathrm{d}\rho} f(\rho) + [A - 2L - 3] f(\rho) = 0. \tag{31}$$

Letting $\xi = \rho^2$, the radial equation (31) is converted into

$$\xi \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} f(\xi) + \left(L + \frac{3}{2} - \xi\right) \frac{\mathrm{d}}{\mathrm{d}\xi} f(\xi) - \frac{1}{4}(2L + 3 - A) f(\xi) = 0. \tag{32}$$

Equation (32) is a confluent hypergeometric equation, and its solution is given in terms of the confluent hypergeometric function

$$f(\xi) = \mathrm{F}\left(\frac{2L + 3 - A}{4}, L + \frac{3}{2}, \xi\right). \tag{33}$$

When $\xi \to \infty$, the confluent hypergeometric function behaves as $\exp(\xi)$, where $u(\rho)$ is exponentially divergent. The regularity conditions imply that bound states that exist only are

$$\frac{1}{4}(2L + 3 - A) = -n_r \quad (n_r = 0, 1, 2 \ldots), \tag{34}$$

where n_r is the radial quantum number. The energy equation for the RNHO potential with the condition of pseudospin symmetry is obtained as

$$(\varepsilon + M)\sqrt{\varepsilon - M} = \sqrt{2M}\left(2n_r + L + \frac{3}{2}\right). \tag{35}$$

The radial wave function is expressed as

$$u(r) \approx (Br)^{L+1} \exp\left(-\frac{1}{2} B^2 r^2\right) \mathrm{F}\left(-n_r, L + \frac{3}{2}, B^2 r^2\right). \tag{36}$$

By using the following known relations

$$\mathrm{L}_n^{\mu}(x) = \frac{\Gamma(n + \mu + 1)}{n!\Gamma(\mu + 1)} \mathrm{F}(-n, \mu + 1, x), \tag{37}$$

$$\int_0^{\infty} x^{\mu} e^{-x} \mathrm{L}_n^{\mu}(x) \mathrm{L}_k^{\mu}(x) \mathrm{d}x = \frac{\Gamma(n + \mu + 1)}{n!} \delta_{nk}, \tag{38}$$

and the orthogonality of radial wave functions,

$$\int_0^{\infty} [R_n(r)]^2 r^2 \mathrm{d}r = 1, \quad [R_n(r) = r^{-1} u_n(r)], \tag{39}$$

the normalized radial wave functions can be written as

$$u_{n_r L}(r) = N_{n_r L} (Br)^{L+1} \exp\left(-\frac{1}{2}\varepsilon^2 r^2\right) \mathrm{L}_{n_r}^{\mu}(B^2 r^2) \quad \left(\mu = L + \frac{1}{2}\right). \tag{40}$$

And the normalization constant $N_{n_r L}$ is determinated as

$$N_{n_r L} = \sqrt{\frac{2Bn_r!}{\Gamma(n_r + L + \frac{3}{2})}}. \tag{41}$$

Finally, we get the normalized spinor wave functions as

$$\psi(\vec{r}) = \begin{pmatrix} \varphi(\vec{r}) \\ \chi(\vec{r}) \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{\mathrm{p}}}{\varepsilon - M} \tilde{\chi}_m \\ \tilde{\chi}_m \end{pmatrix} r^{-1} u_{n_r L}(r) H_{n_3}(\theta) \exp(i\Lambda\phi). \tag{42}$$

3 Discussions

In recent years, considerable efforts have been made to obtain the analytical solution of non-central problems. As we know, the Coulombic ring-shaped potential, or Hartmann potential, is another kind of non-central potentials besides the oscillatory ring-shaped potential. The Hartmann potential results from adding a potential proportional to $(r\sin\theta)^{-2}$ to a Coulomb potential and is presented by Hartmann [32]. Due to the possible applications in quantum chemistry and nuclear physics to describe ring-shaped molecules like benzene and interactions between deformed pairs of nuclei, the Hartmann potential has been studying from both non-relativistic and relativistic quantum mechanics viewpoints since its suggestion, and included the energy of the levels [33, 34], the 'accidental' degeneracy and 'hidden' symmetry [35–37], the continuous state and phase shifts [38], etc. However, no pseudospin symmetry is found in the Hartmann potential and the other Coulombic ring-shaped potentials.

When γ is confined to zero, the RNHO potential reduces to the NHO potential, and when η is confined to zero, the RNHO potential reduces to the RHO potential. From (34) and (35), it can be found the eigenenergies $\varepsilon = \varepsilon(n_r, L)$ for the RNHO potential depend only on n_r and L. If $\tilde{L} = L + 1$ is defined as the pseudo-orbital angular momentum of the RNHO potential, then the splitting between the levels with the quantum numbers $(n_r, \tilde{L})$ and $(n_r - 1, \tilde{L} + 2)$ disappears. The levels with quantum numbers $(n_r, \tilde{L})$ and $(n_r - 1, \tilde{L} + 2)$ are fully degenerate as shown in Fig. 11 in Ref. [24].

Since the right-hand side of (35) is always positive, it is clear that any real solution must be greater than M, otherwise the left-hand side would be an imaginary root. We can prove that (35) has only one real solution, and it has to be such that $\varepsilon - M > 0$ by using the Descartes' rule of signs. We can rearrange (35) as

$$\left(\frac{\varepsilon + M}{M}\right)^3 - 2\left(\frac{\varepsilon + M}{M}\right)^2 = 2\left(\frac{2n_r + L + 3/2}{M}\right)^2. \tag{43}$$

By letting

$$x = \frac{\varepsilon + M}{M}, \qquad a = \frac{\sqrt{2}(2n_r + L + 3/2)}{M}, \tag{44}$$

(43) becomes

$$x^3 - 2x^2 - a^2 = 0. \tag{45}$$

Descartes' rule of signs states that the number of positive real roots of an algebraic equation with real coefficients $a_k x^k + \cdots + a_1 x^1 + a_0 = 0$ is never greater than the number of changes of signs in the sequence $a_k, \ldots, a_1, a_0$ and, if less, then always by an even number. Equation (45) shows the number of changes of signs in the sequence $a_k, \ldots, a_1, a_0$ is only one, then only one solution with $\varepsilon - M > 0$ exists. Therefore, for the given quantum numbers n_r and L, only discrete, positive energies (greater than M) are allowed.

4 Conclusions

Under the condition of pseudospin symmetry, we have solved the Dirac equation with the RNHO potential and obtained the energy equation and spin wave functions for bound states. The algebraic property of the energy equation is also discussed by using Descartes' rule of signs, showing that for the Dirac-RNHO problem only discrete, positive energies are allowed with the given quantum numbers n_r and L.

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